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**N° 2189**
Janvier 1994

PROGRAMME 4

Robotique, image
et
vision

Rapport
de recherche

1994

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SOPHIA ANTIPOLIS

# The planar motion with bounded derivative of the curvature and its suboptimal paths *

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Projet Prisme

Rapport de recherche n° 2189 — Janvier 1994 — 31 pages

**Abstract:** We describe the construction of suboptimal trajectories of the problem of a planar motion with bounded derivative of the curvature and we prove their suboptimality. 'Suboptimal' means longer than the optimal by no more than a constant depending only on the bound $B$ for the curvature's derivative. The initial and final coordinates, curvatures and tangent angles are given. The tangent angle and the curvature of the path are assumed to be continuous. The bound $B$ and the distance $d$ between the initial and final points satisfy an inequality of the kind $d \gg 1/\sqrt{B}$.

**Key-words:** car-like robot, (sub)optimal path, clothoid, Maximum Principle of Pontryagin

*This research was partially supported by the ESPRIT III BRA Project 6546 (PROMotion).
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# Etude du mouvement plan lorsque la dérivée de la courbure est bornée et construction de trajectoires sous-optimales

**Résumé :** Nous décrivons la construction de trajectoires sous-optimales pour un mouvement plan, la dérivée de la courbure étant bornée. La sous-optimalité des trajectoires est démontrée. 'Sous-optimale' signifie que la longueur de la trajectoire n'excéde pas le produit de la longueur optimale par une constante qui dépend de la borne $B$ sur la dérivée de la courbure. Des configurations initiale et finale (position, orientation et courbure) sont données. Les angles tangents et la courbure sont suposés continus sur la trajectoire. La borne $B$ et la distance entre les positions initiale et finale satisfont l'inégalité $d \gg 1/\sqrt{B}$.

**Mots-clé :** robot mobile, chemin (sous)optimal, clothoide, principe du maximum de Pontryagin

# 1 Introduction

We consider the problem of finding the shortest path connecting two given points of the Euclidian plane which has given initial and final tangent angles and initial and final curvatures, whose tangent angle and curvature vary continuously, the speed of changing the curvature being bounded. We consider paths which contain no cusps.

The problem has a real background - this is the problem to find the shortest paths for a car to go from one given point to another with the above mentioned initial and final conditions. One can turn the wheels of a car with a bounded speed. Hence, the speed of changing the curvature of the path of a real car is bounded.

This and similar problems have been the object of several efforts recently. Dubins in [5] considers the problem of constructing the optimal trajectory between two given points with given tangent angles and with bounded curvature (cusps are not allowed). He proves that there exists a unique optimal trajectory which is a concatenation of at most three pieces; every piece is either a straight line segment or an arc of a circle of fixed radius. The same model is considered by Cockayne and Hall in [4] but from another point of view: they provide the classes of trajectories by which a moving "orientated point" can reach a given point in a given direction and they obtain the set of all the points reachable at a fixed time.

Reeds and Shepp in [10] solve a similar problem, when cusps are allowed. They obtain the list of all possible optimal trajectories. This list containes forty eight types of trajectories. Each of them is a finite concatenation of pieces each of which is either a straight line or an arc of a circle.

Laumond and Souères in [7] obtain a complete synthesis for the Reeds-Shepp model in the case without obstacles.

All these authors use very particular methods in their proofs. It seems very difficult to generalize them. That is why the same problem is solved by Sussman and Tang in [11] and by Boissonnat, Cérézo and Leblond in [1] by means of simpler arguments based on the Maximum Principle of Pontryagin. Using these arguments allows to treat more difficult models as the one considered in this paper. Here we consider a similar problem but now with a bounded derivative of the curvature (cusps are not allowed).

The same problem is considered in [2] by Boissonnat, Cérézo and Leblond. It seems to be unknown whether the number of switching points in the optimal trajectory is finite or not (i.e. whether the control functions have finitely many points of discontinuity). That is why we concentrate the attention on the explicit description of suboptimal trajectories (i.e. not more than a constant longer than the optimal one) and of their construction. Two students - P.Cohen and A.Casta - wrote a programme in MAPLE which draws such suboptimal paths. In [6] we consider the problem to construct suboptimal paths in the case when cusps are allowed.

In §2 we consider the theoretical aspect of the problem, using the Maximum Principle of Pontryagin. We obtain that if the optimal trajectory is piecewise regular then it must be a concatenation of arcs of clothoids and of straight line segments. Thus we construct the suboptimal path from such pieces in §4. We prove the suboptimality of the constructed path in §5 by means of some geometric properties of clothoids which are exposed in §3.

## 2 Statement of the problem, existence of an optimal solution and application of the Maximum Principle of Pontryagin to this problem

We study the shortest $C^2$ and piecewise $C^3$ path on the plane joining two given points with given tangent angles and curvatures along which the derivative of the curvature remains bounded. The tangent angle $\alpha(t)$ between the axis $Ox$ and the tangent-vector to the path is a continuous and piecewise $C^2$ function, the curvature $u(t)$ is a continuous and piecewise $C^1$ function.

We have the following system (from now on we denote "$d/dt$" by "$\cdot$"):

$$\dot{X}(t) = \begin{cases} \dot{x}(t) = \cos\alpha(t) \\ \dot{y}(t) = \sin\alpha(t) \\ \dot{\alpha}(t) = u(t) \\ \dot{u}(t) = u'(t) \qquad |u'(t)| \leq B \end{cases} \tag{1}$$

with initial and final conditions:

$$X(0) = (x^0, y^0, \alpha^0, u^0), \qquad X(T) = (x^1, y^1, \alpha^1, u^1) \tag{2}$$

We control the derivative of the curvature by the control function $u'$. The control function $u'$ is a measurable, real-valued function and $u' \in U$, where $U = [-B, +B]$. We want to find such $X(t)$ that the associated control function $u'(t)$ should minimize the length of the path

$$J(u') = T = \int_0^T dt \tag{3}$$

Here the variable $t$ is the arc length but it will be called the time because the point moves with a constant speed 1, that is why this "minimum length problem" is also a "minimum time problem".

The complete controllability of system (1) and the existence of an optimal solution for the problem (1)-(3) is proved in [2].

To obtain necessary conditions for the control function $u'(t)$ and for the trajectory ($x(t)$, $y(t)$, $\alpha(t)$, $u(t)$) to be optimal we can apply Maximum Principle of Pontryagin (see the details in [2]).

A measurable control function $u'$ and the associated trajectory of (1) satisfying all conditions of the Maximum Principle of Pontryagin (see [3], th.5.1i, [9], Chapter 1, th.1 and [2] subsection 3.1) will be called *extremal control* and *extremal trajectory.* A point $X(t_p)$ of an extremal trajectory will be called a *switching point* if at $t = t_p$ the control function $u'(t)$ has a discontinuity.

After applying the Maximum Principle of Pontryagin we obtain the following result (see [2]):

**Lemma 2.1** *If the control function of the extremal path has finitely many points of discontinuity then the extremal path of (1) is the closure of a union of open arcs of clothoids ($u'(t) \equiv \pm B$) on open intervals of $[0,T]$ and line segments in one and the same direction $\varphi$ ($u'(t) \equiv 0$) on open intervals of $[0,T]$.*

A clothoid is a curve along which the curvature $u(t)$ depends linearly on the arc length $t$ and varies continuously from $-\infty$ to $+\infty$. In our case we consider only clothoids which satisfy the following equation (see Lemma 2.1):

$$u(t) = \pm Bt, \qquad t \in (-\infty, +\infty)$$

We can also define the clothoid by its parametrized form (setting $x(0) = y(0) = 0$, $\alpha(0) = 0$, $u(0) = 0$)

$$\begin{cases} x(t) = \sqrt{2/B}\int_0^{t\sqrt{B/2}}\cos(\tau^2)d\tau \\ y(t) = \pm\sqrt{2/B}\int_0^{t\sqrt{B/2}}\sin(\tau^2)d\tau \end{cases}$$

The two possible choices of the sign correspond to the two possible orientations of the clothoid.

It is not clear whether the optimal path is regular or not and how to compute the optimal path explicitly. That is why in the present paper we shall construct in §4 a suboptimal path explicitly in the case when the distance between the initial and the final point is much greater than $1/\sqrt{B}$ (the exact definition is given in §4).

The suboptimal path consists of a line segment and of four pieces of a clothoid, its curvature and tangent angle are continuous, it has four switching points, see §4.

In order to prove the suboptimality of the path constructed in §4, i.e. that it is no more than a fixed constant (depending only on $B$) longer than the optimal one we prove some geometric properties of clothoids in §3. The suboptimality is proved in §5.

# 3 Geometric properties of the clothoid.

## 3.1 Properties of an individual clothoid.

Consider a half-clothoid

$$\begin{cases} \dot{x}(t) = \cos(Bt^2/2) & x(0) = 0 & t \geq 0 \\ \dot{y}(t) = \sin(Bt^2/2) & y(0) = 0 & B > 0 \end{cases} \tag{4}$$

Call $B$ "the parameter of the clothoid" and set $B = 2$ for simplicity. Thus we consider the half-clothoid

$$\begin{cases} \dot{x} = \cos t^2 & x(0) = 0 & t \geq 0 \\ \dot{y} = \sin t^2 & y(0) = 0 \end{cases} \tag{5}$$

**Remark 3.1** *A half-clothoid of the opposite orientation is defined by equations*

$$\begin{cases} \dot{x}(t) = \cos(-Bt^2/2) & x(0) = 0 & t \geq 0 \\ \dot{y}(t) = \sin(-Bt^2/2) & y(0) = 0 & B > 0 \end{cases}$$

Fix an angle $\alpha_* \geq 0$. Let $P_1, P_2, \ldots$ denote the consecutive points on the half-clothoid where the tangent line has direction $\alpha_*$ (mod$\pi$, not mod$2\pi$, $t_1 < t_2 < \ldots$). Set $P_i = (x_i, y_i)$, $x_i = x(t_i)$, $y_i = y(t_i)$ (see Figure 1).

**Proposition 3.2** *$\widehat{P_1P_2}$ is the longest among the arcs $\widehat{P_iP_{i+1}}$. Its length depends continuously and monotonously on the choice of the angle $\alpha_*$.*

**Proof.**

$$|\widehat{P_iP_{i+1}}| = \int_{\sqrt{\alpha_*+(i-1)\pi}}^{\sqrt{\alpha_*+i\pi}} \sqrt{\cos^2 t^2 + \sin^2 t^2}dt = \sqrt{\alpha_* + i\pi} - \sqrt{\alpha_* + (i-1)\pi} =$$

$$= \frac{\pi}{\sqrt{\alpha_* + i\pi} + \sqrt{\alpha_* + (i-1)\pi}}$$

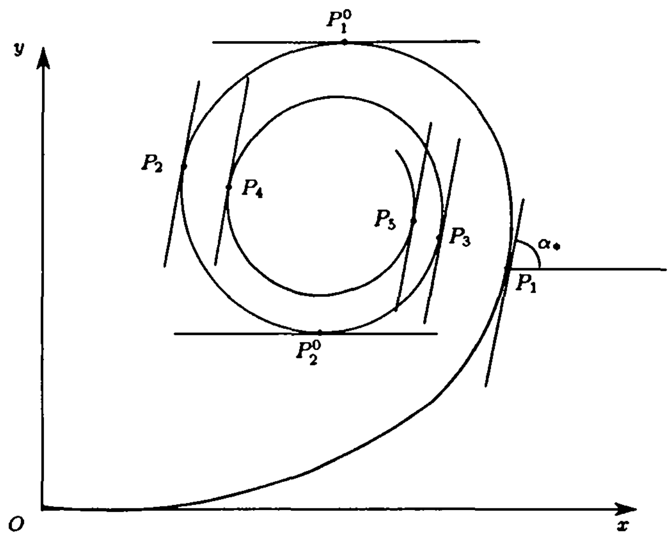



Figure 1

Both statements follow directly from these equalities. □

Define as "the centre of the half-clothoid" the point $O_c$ with coordinates $(x_{O_c}, y_{O_c})$ defined as follows:

$$\begin{cases} x_{O_c} = \int_0^\infty \cos\tau^2 d\tau \\ y_{O_c} = \int_0^\infty \sin\tau^2 d\tau \end{cases}$$

Consider the coordinate system with the centre at the centre $O_c$ of the half-clothoid and with the axes $O_c x_c$, $O_c y_c$ parallel to the corresponding axes of the coordinate system $Oxy$ (see Figure 2). In the coordinate system $O_c x_c y_c$ the coordinates of the point $(x_c, y_c)$ of the clothoid (5) are defined by the formulas:

$$\begin{cases} x_c(t) = x(t) - x_{O_c} = -\int_t^\infty \cos\tau^2 d\tau \\ y_c(t) = y(t) - y_{O_c} = -\int_t^\infty \sin\tau^2 d\tau \end{cases} \tag{6}$$

Denote by $\vec{\rho}$ the radius-vector of a point of the half-clothoid in the coordinate system $O_c x_c y_c$. Then

$$\rho^2 = {x_c}^2 + {y_c}^2$$

and

$$\dot{\rho}(t) = \frac{1}{\rho}(x_c\dot{x}_c + y_c\dot{y}_c) =$$

$$= \frac{1}{\rho}(-\cos t^2 \int_t^\infty \cos\tau^2 d\tau - \sin t^2 \int_t^\infty \sin\tau^2 d\tau) =$$

$$= -\frac{1}{\rho}\int_t^\infty \cos(\tau^2 - t^2)d\tau = -\frac{1}{\rho}\int_{t^2}^\infty \frac{\cos(\eta - t^2)d\eta}{2\sqrt{\eta}} = -\frac{1}{\rho}\int_0^\infty \frac{\cos\nu d\nu}{2\sqrt{\nu + t^2}}$$

Thus

$$\dot{\rho}(t) = -\frac{1}{2\rho}\int_0^\infty \frac{\cos\tau d\tau}{\sqrt{\tau + t^2}} \tag{7}$$

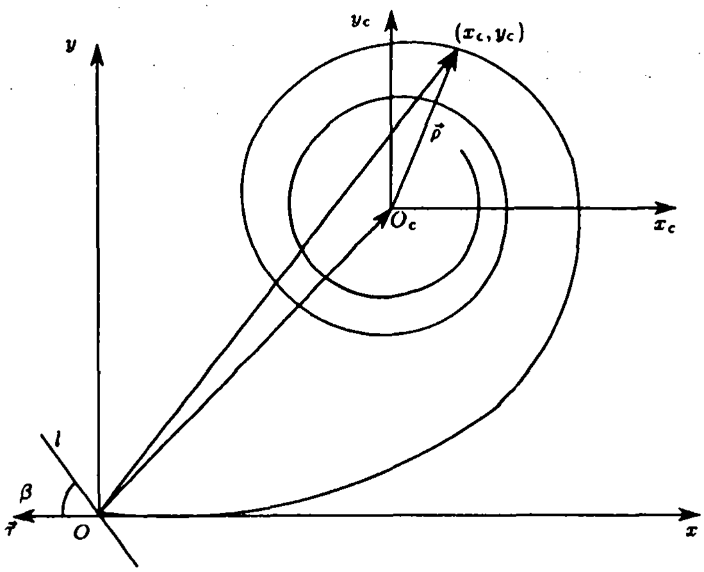



Figure 2

**Lemma 3.3** *The length of the radius-vector $\vec{\rho}(t)$ of the clothoid defined by system (5) is a monotonously decreasing function of $t$:*

$$\dot{\rho} < 0$$

**Proof**

Set

$$t^2 = a, \qquad \int_0^\infty \frac{\cos\tau d\tau}{\sqrt{\tau + a}} = I(a)$$

The function $\cos\tau$ is periodic with period $2\pi$. So using the property of the symmetry of the function $\cos\tau$ ($\cos(\pi-\tau) = \cos(\pi+\tau) = -\cos\tau$, $\cos(2\pi-\tau) = \cos\tau$) we can consider instead of the integral $I(a)$ the following integral:

$$\int_0^{\pi/2} \Sigma \cos\tau d\tau$$

where

$$\Sigma = \sum_{k=0}^{\infty} \left( \frac{1}{\sqrt{a+\tau+2k\pi}} - \frac{1}{\sqrt{\pi-\tau+a+2k\pi}} - \frac{1}{\sqrt{\pi+\tau+a+2k\pi}} + \frac{1}{\sqrt{2\pi-\tau+a+2k\pi}} \right)$$

This series is convergent because

$$\frac{1}{\sqrt{a+\tau+2k\pi}} - \frac{1}{\sqrt{\pi-\tau+a+2k\pi}} =$$

$$= \frac{\pi - 2\tau}{\sqrt{a+\tau+2k\pi}\sqrt{\pi-\tau+a+2k\pi}(\sqrt{a+\tau+2k\pi} + \sqrt{\pi-\tau+a+2k\pi})} = O\left(\frac{1}{k\sqrt{k}}\right)$$

and

$$-\frac{1}{\sqrt{\pi+\tau+a+2k\pi}} + \frac{1}{\sqrt{2\pi-\tau+a+2k\pi}} =$$

$$\frac{-\pi+2\tau}{\sqrt{\pi+\tau+a+2k\pi}\sqrt{2\pi-\tau+a+2k\pi}(\sqrt{\pi+\tau+a+2k\pi}+\sqrt{2\pi-\tau+a+2k\pi})}=O\left(\frac{1}{k\sqrt{k}}\right)$$

Consider the first four terms of the series. The function $f(\xi)=1/\sqrt{\xi}$ is convex and monotonously decreasing, see Figure 3. For the middle lines $KM$ and $LM$ of the trapezoids $EABF$ and $GCDH$ respectively we have $LM\subset KM$.

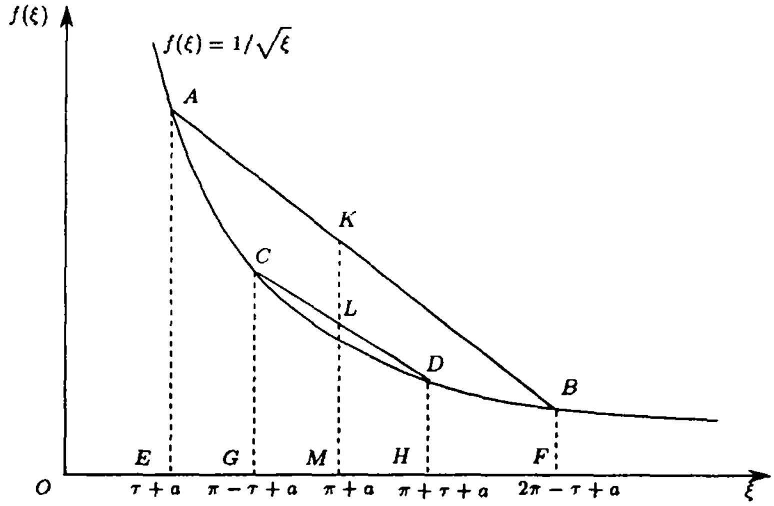



Figure 3

We have the followings formulas:

$$\frac{1}{\sqrt{\tau+a}}+\frac{1}{\sqrt{2\pi-\tau+a}}=2|KM|,$$

$$\frac{1}{\sqrt{\pi-\tau+a}}+\frac{1}{\sqrt{\pi+\tau+a}}=2|LM|,$$

$$|LM|<|KM|$$

Hence

$$\frac{1}{\sqrt{\tau+a}}-\frac{1}{\sqrt{\pi-\tau+a}}-\frac{1}{\sqrt{\pi+\tau+a}}+\frac{1}{\sqrt{2\pi-\tau+a}}>0$$

Every following sum of four terms in the series can be considered analogously. This proves that the sum of the series under consideration is positive. The function $\cos\tau, \tau\in[0,\pi/2]$ is non-negative. Hence, the integral $I(a)$ is positive and the derivative of the length of the radius-vector $\vec{\rho}(t)$ is negative.

The lemma is proved. □

**Lemma 3.4** *The derivative of the length of the radius-vector $\vec{\rho}(t)$ of the clothoid defined by system (5) is a monotonously increasing function of $t$, i.e.*

$$\ddot{\rho} > 0 \tag{8}$$

The lemma is proved in subsection **3.2**.

We give a geometric interpretation of the inequality $\ddot{\rho} > 0$. Denote by $\gamma(t)$ the angle between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$ of the point of the clothoid (5). We have

$$\dot{\rho} = \cos\gamma \tag{9}$$

The angle $\gamma$ is in the interval $(\pi/2, \pi)(\mathrm{mod}2\pi)$ (because $\dot{\rho} < 0$, see Lemma 3.3). Hence, the function $\sin\gamma$ is positive. We have

$$\ddot{\rho} = -\dot{\gamma}\sin\gamma \tag{10}$$

and obtain, from (8), that

$$\dot{\gamma} < 0 \tag{11}$$

So we obtain the geometric interpretation of Lemma 3.4:

**Remark 3.5** *The angle $\gamma(t)$ between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$ is a monotonously decreasing function of $t$.*

**Corollary 3.6** *If we have an "unwinding" half-clothoid (i.e. half-clothoid with decreasing absolute value of the curvature) defined by the equations:*

$$\begin{cases} x(t) = \int_0^t \cos(\tau^2 + u_0\tau + \alpha_0)d\tau & x(0) = x_0 \quad u_0 < 0 \\ y(t) = \int_0^t \sin(\tau^2 + u_0\tau + \alpha_0)d\tau & y(0) = y_0 \quad t \geq 0 \end{cases}$$

*then for such a clothoid we have the following conditions:*

$$\dot{\rho} > 0$$

$$\ddot{\rho} > 0$$

**Corollary 3.7** *If two half-clothoids $clA$ and $clB$ have the same centre $O_c$, the same orientation and the same parameter $B$ then either they coincide or they have no common point.*

*Consider the circle $C$ with centre at the centre $O_c$ of $clA$ and with radius equal to the distance between the centre of $clA$ and its point of zero curvature. Denote by $\partial C$ the circumference with centre at $O_c$ and with the same radius. Then $C \setminus O_c$ is the union of non-intersecting half-clothoids. The mapping which maps each half-clothoid on its point of zero curvature (lying on $\partial C$) is a bijection from the set of half-clothoids onto $\partial C$.*

**Proof** If $clA$ and $clB$ intersect, then at the intersection point they have equal radius-vectors, hence, equal curvatures (see Lemma 3.3), hence, equal values of $\dot{\rho}$ (see Lemma 3.4), hence, they must coincide, because they are obtained by integrating the equations $\dot{x} = \cos(t - t_0)^2$, $\dot{y} = \sin(t - t_0)^2$ with equal initial data $(x_0, y_0, t_0)$. □

### 3.2 Proof of Lemma 3.4.

An arbitrary point $A$ of the clothoid (5) has a tangent vector $\vec{\tau}(t)$ with coordinates $(\cos t^2, \sin t^2)$ (see Figure 4). Consider a point $D$ of the clothoid (5) with tangent vector $\vec{\tau}_n = (0,1)$. The point $A$ is mapped onto the point $D$ by means of the rotation on angle $\theta$ defined by the rotation matrix

$$\begin{pmatrix} \sin t^2 & -\cos t^2 \\ \cos t^2 & \sin t^2 \end{pmatrix}$$

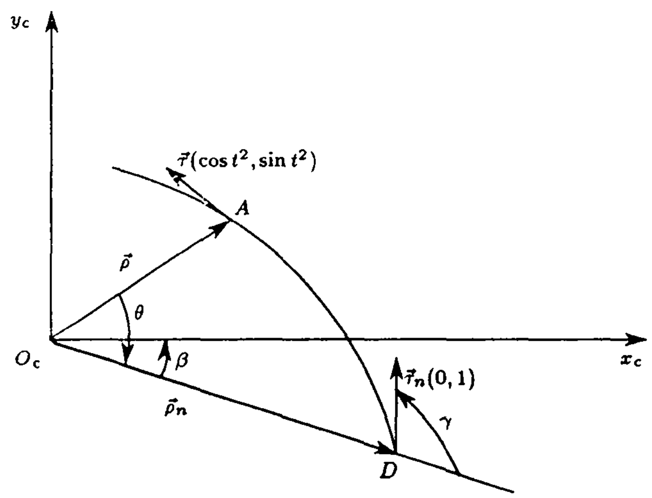



Figure 4

Hence, the radius-vector $\vec{\rho} = (-\int_t^\infty \cos\tau^2 d\tau, -\int_t^\infty \sin\tau^2 d\tau)$ (see (6)) is mapped into the radius-vector

$$\vec{\rho}_n = \left(-\sin t^2 \int_t^\infty \cos\tau^2 d\tau + \cos t^2 \int_t^\infty \sin\tau^2 d\tau \quad , \right.$$

$$\left. -\cos t^2 \int_t^\infty \cos\tau^2 d\tau - \sin t^2 \int_t^\infty \sin\tau^2 d\tau \right) =$$

$$= \left(\int_t^\infty \sin(\tau^2 - t^2) d\tau \quad , \quad -\int_t^\infty \cos(\tau^2 - t^2) d\tau\right) =$$

$$= \left(\int_0^\infty \frac{\sin\nu d\nu}{2\sqrt{\nu + t^2}} \quad , \quad -\int_0^\infty \frac{\cos\nu d\nu}{2\sqrt{\nu + t^2}}\right)$$

We want to investigate the function $d\rho/dt$. Instead of it we can investigate the function $d\gamma/dt$ (see formula (9)). Denote by $\beta$ the angle between the vector $\vec{\rho}_n$ and the axis $O_c x_c$. At the point $D$ we have the following relations between the angles $\gamma, \beta$ and the coordinates $x_n, y_n$ of the vector $\vec{\rho}_n$:

$$\cot\gamma = -\tan\beta = -\frac{y_n}{x_n} = \int_0^\infty \frac{\cos\nu d\nu}{2\sqrt{\nu + t^2}} \Big/ \int_0^\infty \frac{\sin\nu d\nu}{2\sqrt{\nu + t^2}}$$

Compute the derivative $d(\tan\beta)/dt$:

$$\frac{d(\tan\beta)}{dt} = -\frac{t}{4x_n{}^2}\left[\int_0^\infty \frac{\cos\tau d\tau}{(\sqrt{\tau+t^2})^3}\int_0^\infty \frac{\sin\tau d\tau}{\sqrt{\tau+t^2}} - \int_0^\infty \frac{\sin\tau d\tau}{(\sqrt{\tau+t^2})^3}\int_0^\infty \frac{\cos\tau d\tau}{\sqrt{\tau+t^2}}\right] =$$

$$= -\frac{t}{4x_n{}^2}\left[\left\{\frac{3}{2}\int_0^\infty \frac{\sin\tau d\tau}{(\sqrt{\tau+t^2})^5} - \frac{\sin\tau}{(\sqrt{\tau+t^2})^3}\bigg|_0^\infty\right\}\int_0^\infty \frac{\sin\tau d\tau}{\sqrt{\tau+t^2}} - \right.$$

$$\left. -\left\{\frac{1}{2}\int_0^\infty \frac{\sin\tau d\tau}{(\sqrt{\tau+t^2})^3} - \frac{\sin\tau}{\sqrt{\tau+t^2}}\bigg|_0^\infty\right\}\int_0^\infty \frac{\sin\tau d\tau}{(\sqrt{\tau+t^2})^3}\right] =$$

$$= -\frac{t}{8x_n{}^2}\left[3\int_0^\infty \frac{\sin\tau d\tau}{(\sqrt{\tau+t^2})^5}\int_0^\infty \frac{\sin\tau d\tau}{\sqrt{\tau+t^2}} - \left(\int_0^\infty \frac{\sin\tau d\tau}{(\sqrt{\tau+t^2})^3}\right)^2\right]$$

(We use integration by parts.)

Denote the expression in the brackets as $J(t^2)$. Consider $J(t^2)$ with $\infty$ changed to $2\pi p$ ($p \in \mathrm{N}, p > 1$). Consider the corresponding Riemann sums with step $\Delta = \pi/n$ instead of the integrals:

$$\int_0^{2\pi p} \frac{\sin\tau d\tau}{(\sqrt{\tau+t^2})^i} \cong \sum_{k=1}^{2np} \frac{\sin\tau_k}{(\sqrt{\tau_k+t^2})^i}\Delta + O(\Delta), \quad \tau_k = \pi k/n, \quad i = \{1,3,5\} \tag{12}$$

The function $\sin\tau$ is periodic with period $2\pi$ and $\sin(\pi+\tau) = -\sin\tau$.

Denote the three Riemann sums (corresponding to the three integrals) by

$$d_1 + \ldots + d_{np}, \qquad g_1 + \ldots + g_{np}, \qquad h_1 + \ldots + h_{np}$$

where if $j = s + \nu n$, $s = 1, \ldots, n$, $\nu = 0, \ldots, p-1$, then

$$d_j = \frac{\sin\tau_s}{\sqrt{\tau_s + 2\nu\pi + t^2}} - \frac{\sin\tau_s}{\sqrt{\tau_s + 2\nu\pi + \pi + t^2}},$$

$$g_j = \frac{\sin\tau_s}{(\sqrt{\tau_s + 2\nu\pi + t^2})^3} - \frac{\sin\tau_s}{(\sqrt{\tau_s + 2\nu\pi + \pi + t^2})^3}$$

$$h_j = \frac{\sin\tau_s}{(\sqrt{\tau_s + 2\nu\pi + t^2})^5} - \frac{\sin\tau_s}{(\sqrt{\tau_s + 2\nu\pi + \pi + t^2})^5},$$

Show that

$$I \equiv 3d_jh_j - g_j{}^2 \geq 0 \tag{13}$$

Set $\tau_s + 2\nu\pi = a$. Then rewrite $I$ as follows:

$$3\left(\frac{\sin a}{(\sqrt{a+t^2})^5} - \frac{\sin a}{(\sqrt{a+\pi+t^2})^5}\right)\left(\frac{\sin a}{\sqrt{a+t^2}} - \frac{\sin a}{\sqrt{a+\pi+t^2}}\right) - \left(\frac{\sin a}{(\sqrt{a+t^2})^3} - \frac{\sin a}{(\sqrt{a+\pi+t^2})^3}\right)^2$$

Denote $\sqrt{a+t^2}$ by $\alpha$, $\sqrt{a+\pi+t^2}$ by $\beta$. Then

$$I = \sin^2 a\left[3\left(\frac{1}{\alpha^5}-\frac{1}{\beta^5}\right)\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)-\left(\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)^2\right] =$$

$$= \sin^2 a\left[3\frac{(\beta^5-\alpha^5)(\beta-\alpha)}{\alpha^6\beta^6}-\frac{(\beta^3-\alpha^3)^2}{\alpha^6\beta^6}\right] =$$

$$= \sin^2 a\left[\frac{3(\beta-\alpha)^2(\beta^4+\beta^3\alpha+\beta^2\alpha^2+\beta\alpha^3+\alpha^4)-(\beta-\alpha)^2(\beta^2+\beta\alpha+\alpha^2)^2}{\alpha^6\beta^6}\right] =$$

$$= \sin^2 a\left[\frac{(\beta^2-\alpha^2)^2[3(\beta^4+\beta^3\alpha+\beta^2\alpha^2+\beta\alpha^3+\alpha^4)-(\beta^2+\beta\alpha+\alpha^2)^2]}{\alpha^6\beta^6(\beta+\alpha)^2}\right] =$$

$$= \sin^2 a\left[\frac{\pi^2(2\beta^4+2\alpha^4+\beta^3\alpha+\beta\alpha^3)}{\alpha^6\beta^6(\beta+\alpha)^2}\right] \geq 0$$

Thus we prove (13). Show that

$$K \equiv 3(d_ih_j+d_jh_i)-2g_ig_j \geq 0 \tag{14}$$

Set

$$\tau_s+2\nu\pi = a_i, \qquad \tau_w+2\nu\pi = a_j,$$

$$\sqrt{a_i+t^2} = \alpha, \qquad \sqrt{a_i+\pi+t^2} = \beta,$$

$$\sqrt{a_j+t^2} = \gamma, \qquad \sqrt{a_j+\pi+t^2} = \delta.$$

Rewrite $K$ as follows:

$$K = 3\left[\left(\frac{\sin a_i}{\alpha^5}-\frac{\sin a_i}{\beta^5}\right)\left(\frac{\sin a_j}{\gamma}-\frac{\sin a_j}{\delta}\right)+\left(\frac{\sin a_j}{\gamma^5}-\frac{\sin a_j}{\delta^5}\right)\left(\frac{\sin a_i}{\alpha}-\frac{\sin a_i}{\beta}\right)\right]-$$

$$-2\left(\frac{\sin a_i}{\alpha^3}-\frac{\sin a_i}{\beta^3}\right)\left(\frac{\sin a_j}{\gamma^3}-\frac{\sin a_j}{\delta^3}\right) =$$

$$= \frac{\pi^2\sin a_i\sin a_j}{(\beta+\alpha)(\gamma+\delta)\alpha\beta\gamma\delta}\left[\frac{3(\beta^4+\beta^3\alpha+\beta^2\alpha^2+\beta\alpha^3+\alpha^4)}{\alpha^4\beta^4}+\frac{3(\delta^4+\delta^3\gamma+\delta^2\gamma^2+\delta\gamma^3+\gamma^4)}{\gamma^4\delta^4}-\right.$$

$$\left.-\frac{2(\beta^2\delta^2+\beta^2\delta\gamma+\beta^2\gamma^2+\delta^2\beta\alpha+\beta\alpha\delta\gamma+\gamma^2\beta\alpha+\alpha^2\delta^2+\alpha^2\delta\gamma+\alpha^2\gamma^2)}{\alpha^2\beta^2\gamma^2\delta^2}\right] \tag{15}$$

Estimate the expression in the brackets (denote it by $L$).

$$L = \left\{ \frac{\alpha^4 + 2\alpha^2\beta^2 + \beta^4}{\alpha^4\beta^4} + \frac{\gamma^4 + 2\gamma^2\delta^2 + \delta^4}{\gamma^4\delta^4} - \frac{2(\beta^2\delta^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2)}{\alpha^2\beta^2\gamma^2\delta^2} \right\} +$$

$$+ \left\{ \frac{1}{2}\left( \frac{1}{\alpha^3\beta} + \frac{1}{\alpha\beta^3} + \frac{1}{\gamma^3\delta} + \frac{1}{\gamma\delta^3} \right) - \frac{2}{\alpha\beta\gamma\delta} \right\} +$$

$$+ \left\{ 2\left( \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4} \right) - \frac{2(\beta^2\delta\gamma + \delta^2\beta\alpha + \gamma^2\beta\alpha + \alpha^2\delta\gamma)}{\alpha^2\beta^2\gamma^2\delta^2} \right\} +$$

$$+ \left\{ \frac{1}{\alpha^2\beta^2} + \frac{1}{2}\left( \frac{1}{\alpha^3\beta} + \frac{1}{\alpha\beta^3} + \frac{1}{\gamma^3\delta} + \frac{1}{\gamma\delta^3} \right) + \frac{1}{\gamma^2\delta^2} \right\} \tag{16}$$

The expression in the first parentheses is positive. Really,

$$\frac{\alpha^4 + 2\alpha^2\beta^2 + \beta^4}{\alpha^4\beta^4} + \frac{\gamma^4 + 2\gamma^2\delta^2 + \delta^4}{\gamma^4\delta^4} = \left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right)^2 + \left( \frac{1}{\gamma^2} + \frac{1}{\delta^2} \right)^2 >$$

$$> 2\left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right)\left( \frac{1}{\gamma^2} + \frac{1}{\delta^2} \right) = \frac{2(\beta^2\delta^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2)}{\alpha^2\beta^2\gamma^2\delta^2}$$

The expression within the second parentheses is also positive because we have the following inequality:

$$\frac{1}{4}\left\{ \frac{1}{\alpha^3\beta} + \frac{1}{\alpha\beta^3} + \frac{1}{\gamma^3\delta} + \frac{1}{\gamma\delta^3} \right\} > \left( \frac{1}{\alpha^3\beta\alpha\beta^3\gamma^3\delta\gamma\delta^3} \right)^{1/4} = \frac{1}{\alpha\beta\gamma\delta}$$

Hence

$$\frac{1}{2}\left\{ \frac{1}{\alpha^3\beta} + \frac{1}{\alpha\beta^3} + \frac{1}{\gamma^3\delta} + \frac{1}{\gamma\delta^3} \right\} > \frac{2}{\alpha\beta\gamma\delta}$$

Estimate the expression within the third parentheses. Denote by $M$ the following fraction:

$$M = \frac{2(\beta^2\delta\gamma + \delta^2\beta\alpha + \gamma^2\beta\alpha + \alpha^2\delta\gamma)}{\alpha^2\beta^2\gamma^2\delta^2} = \frac{2}{\gamma\delta}\left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) + \frac{2}{\alpha\beta}\left( \frac{1}{\gamma^2} + \frac{1}{\delta^2} \right)$$

Using the inequalities

$$\frac{2}{\alpha\beta} < \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

and

$$\frac{2}{\gamma\delta} < \frac{1}{\gamma^2} + \frac{1}{\delta^2}$$

we obtain

$$M < 2\left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right)\left( \frac{1}{\gamma^2} + \frac{1}{\delta^2} \right) < \left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right)^2 + \left( \frac{1}{\gamma^2} + \frac{1}{\delta^2} \right)^2 =$$

$$= \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{2}{\alpha^2\beta^2} + \frac{1}{\gamma^4} + \frac{1}{\delta^4} + \frac{2}{\gamma^2\delta^2} < 2\left(\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}\right)$$

So the expression within the third parentheses is positive and $L$ (see (16)) is positive. Hence the expression $K$ (see (15)) is non-negative because the points $a_i$, $a_j$ belong to the interval $(0,\pi]$ and, hence, the functions $\sin a_i$, $\sin a_j$ are non-negative. So we prove (14).

From (13) and (14) when $n \to \infty$ it follows that

$$3\int_0^{2\pi p} \frac{\sin \tau d\tau}{(\sqrt{\tau + t^2})^5} \int_0^{2\pi p} \frac{\sin \tau d\tau}{\sqrt{\tau + t^2}} - \left(\int_0^{2\pi p} \frac{\sin \tau d\tau}{(\sqrt{\tau + t^2})^3}\right)^2 > 0$$

If $2\pi p \to \infty$ and $n \to \infty$ we obtain that $J(t^2) > 0$ and, hence, $d(\tan\beta)/dt < 0$. Remember that $\tan\beta = -\cot\gamma$ and $\ddot{\rho} = -\dot{\gamma}\sin\gamma$ (see (10)). Hence,

$$\frac{d(\cot\gamma)}{dt} = -\frac{\dot{\gamma}}{\sin^2\gamma} > 0, \qquad \dot{\gamma} < 0$$

and

$$\ddot{\rho} > 0.$$

The lemma is proved. □

### 3.3 Properties of two arcs of clothoids at their concatenation point.

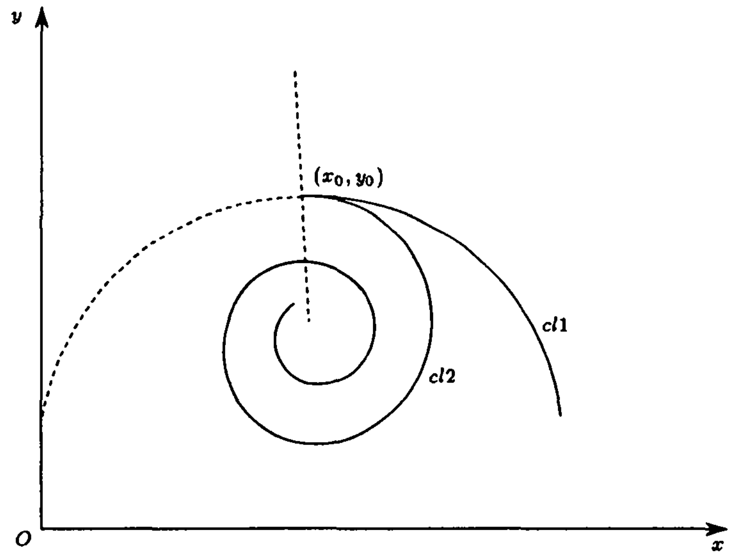



Figure 5

Consider two clothoids $cl1$ and $cl2$ (see Figure 5) which for $t = 0$ have the same initial conditions $(x_0, y_0, \alpha_0, u_0)$, $u_0 < 0$, the absolute value of the curvature of $cl1$ is decreasing with $t$, the one of $cl2$ is increasing with $t$; $cl1$ and $cl2$ are defined by equations:

$$cl1: \quad \begin{cases} x(t) = \int_0^t \cos(\tau^2 + u_0\tau + \alpha_0)d\tau + x_0 \\ y(t) = \int_0^t \sin(\tau^2 + u_0\tau + \alpha_0)d\tau + y_0 \end{cases} \tag{17}$$

$$cl2: \begin{cases} x(t) = \int_0^t \cos(-\tau^2 + u_0\tau + \alpha_0)d\tau + x_0 \\ y(t) = \int_0^t \sin(-\tau^2 + u_0\tau + \alpha_0)d\tau + y_0 \end{cases} \tag{18}$$

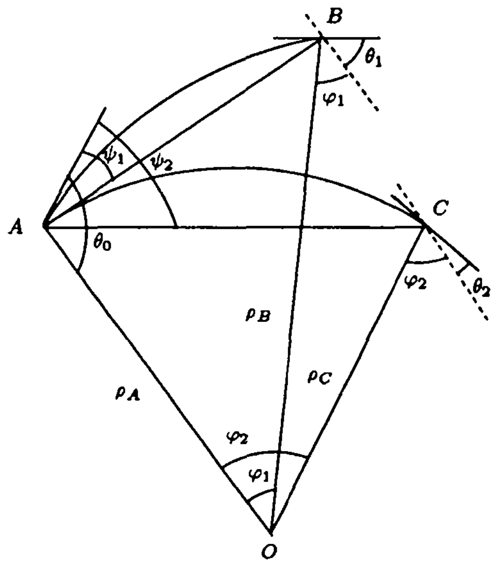



Figure 6

Consider clothoids $cl1$ and $cl2$ on a small interval $t \in [0, s]$ (see Figure 6). On this figure the point $O$ is the centre of $cl1$, the point $A$ is the initial point, the points $B$ and $C$ belong to the clothoids $cl1$ and $cl2$ respectively and $|\widehat{AB}| = |\widehat{AC}| = s$. The angle between the tangent vector to $cl1$ and $cl2$ at point $A$ and the vector equal to $(-\vec{\rho}_A)$ ( $\vec{\rho}_A$ is the radius-vector at point $A$) is denoted by $\theta_0$. The angles between the tangent vectors to $cl1$ and $cl2$ at the points $B$ and $C$ and the vector equal to $(-\vec{\rho}_A)$ are denoted $\theta_1$ and $\theta_2$ respectively. The angles between the tangent vector at the point $A$ and the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are denoted $\psi_1$ and $\psi_2$ respectively. And the angles between the radius-vector $\vec{\rho}_A$ and the radius-vectors $\vec{\rho}_B$ and $\vec{\rho}_C$ at the points $B$ and $C$ are denoted $\varphi_1$ and $\varphi_2$ respectively. Denote by $\delta_i$ $(i = 1, 2)$ the angles between the tangent lines at the points $B$ and $C$ and their radius-vectors $(\delta_i = \theta_i + \varphi_i,\ i = 1, 2)$.

**Lemma 3.8** *For the clothoids $cl1$ and $cl2$ on a small interval $t \in [0, s]$ the following equalities hold:*

$$\rho_B^2 - \rho_C^2 = \frac{4}{3}\rho_A \sin\theta_0 s^3 + O(s^4) \tag{19}$$

$$\delta_1 - \delta_2 = 2s^2 + \frac{2\cos\theta_0}{3\rho_A}s^3 + O(s^4) \tag{20}$$

This lemma is proved in subsection **3.4**.

**Corollary 3.9** *Denote by $C_c$ the point of the clothoid $cl1$ with the same curvature as the point $C$ belonging to clothoid $cl2$. Denote by $C_\rho$ the point of the clothoid $cl1$ with the same length of the radius-vector $\vec{\rho}_C$ as the point $C$ of the clothoid $cl2$; and denote by $C_\gamma$ the point of the clothoid* *cl*1 *with the same angle* $\gamma$ *between the radius-vector and the tangent vector as the point* $C$ *of cl*2. *Denote by* $\gamma_A$, $\gamma_B$, $\gamma_C$ *the angles* $\gamma$ *at the points* $A$, $B$, $C$. *Then the points* $C_c$, $A$, $C_\gamma$, $C_\rho$, $B$ *on a small interval* $[0, s]$ *are encountered in their order along cl*1.

This corollary is proved in subsection **3.4**.

## 3.4 Proof of Lemma 3.8 and of Corollary 3.9.

**Proof of Lemma 3.8.**

Consider a coordinate system $A\xi\eta$ (see Figure 7), the axis $\eta$ coincides with the tangent vector to *cl*1 and *cl*2 at the point $A$, the axis $\xi$ is a perpendicular to the axis $\eta$. In this coordinate system *cl*1 and *cl*2 are defined by the following equations:

$$cl1: \begin{cases} \xi(t) = \int_0^t \cos(\tau^2 + u_0\tau + \pi/2)d\tau \\ \eta(t) = \int_0^t \sin(\tau^2 + u_0\tau + \pi/2)d\tau \end{cases}$$

$$cl2: \begin{cases} \xi(t) = \int_0^t \cos(-\tau^2 + u_0\tau + \pi/2)d\tau \\ \eta(t) = \int_0^t \sin(-\tau^2 + u_0\tau + \pi/2)d\tau \end{cases}$$

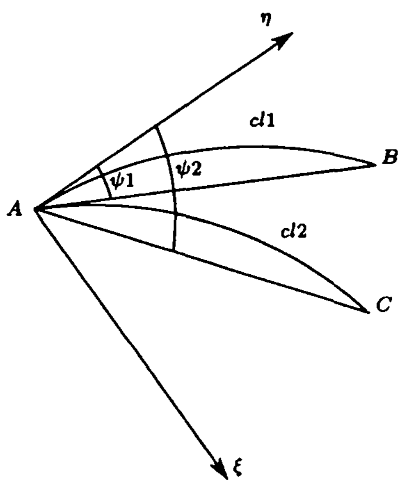



Figure 7

So for the coordinates of the points $B$ and $C$ we have the following formulas:

$$\xi_B(s) = -\int_0^s \sin(\tau^2 + u_0\tau)d\tau \qquad \eta_B(s) = \int_0^s \cos(\tau^2 + u_0\tau)d\tau$$

$$\xi_C(s) = -\int_0^s \sin(-\tau^2 + u_0\tau)d\tau \qquad \eta_C(s) = \int_0^s \cos(-\tau^2 + u_0\tau)d\tau$$

Then, using the Taylor series at 0 for the functions $\sin x, \cos x$ we obtain:

$$\xi_B(s) = -\frac{u_0}{2}s^2 - \frac{1}{3}s^3 + O(s^4) \qquad \eta_B(s) = s - \frac{u_0^2}{6}s^3 + O(s^4)$$

$$\xi_C(s) = -\frac{u_0}{2}s^2 + \frac{1}{3}s^3 + O(s^4) \qquad \eta_C(s) = s - \frac{u_0^2}{6}s^3 + O(s^4)$$

But $\tan\psi_1 = \xi_B(s)/\eta_B(s)$ and $\tan\psi_2 = \xi_C(s)/\eta_C(s)$. Now we use Taylor series again and obtain the following formulas for $\tan\psi_1$ and $\tan\psi_2$.

$$\tan\psi_1 = -\frac{u_0}{2}s - \frac{1}{3}s^2 + O(s^3) \qquad \tan\psi_2 = -\frac{u_0}{2}s + \frac{1}{3}s^2 + O(s^3)$$

Since we consider the clothoids $cl1$ and $cl2$ on a small interval $[0, s]$, we can use for the angles $\psi_1$ and $\psi_2$ the following formulas:

$$\psi_1 = -\frac{u_0}{2}s - \frac{1}{3}s^2 + O(s^3) \qquad \psi_2 = -\frac{u_0}{2}s + \frac{1}{3}s^2 + O(s^3) \tag{21}$$

Compute the values of $\rho_B^2$ and $\rho_C^2$. For this purpose we use the cosine theorem, the Taylor series and formulas (21):

$$\rho_B^2 = \rho_A^2 + s^2 - 2\rho_A s\cos(\theta_0 - \psi_1) =$$

$$= \rho_A^2 + s^2 - 2\rho_A s\left[\cos\theta_0\cos\left(-\frac{u_0}{2}s - \frac{1}{3}s^2 + O(s^3)\right) + \sin\theta_0\sin\left(-\frac{u^0}{2}s - \frac{1}{3}s^2 + 0(s^3)\right)\right] =$$

$$= \rho_A^2 + s^2 - 2\rho_A s\left[\cos\theta_0\left(1 - \frac{1}{2}\left(\frac{u_0}{2}s + \frac{1}{3}s^2\right)^2\right) - \sin\theta_0\left(\frac{u_0}{2}s + \frac{1}{3}s^2\right)\right] =$$

$$= \rho_A^2 - 2\rho_A\cos\theta_0 s + (1 + \rho_A u_0\sin\theta_0)s^2 + \left(\rho_A\frac{u_0^2}{4}\cos\theta_0 + \frac{2}{3}\rho_A\sin\theta_0\right)s^3 + O(s^4);$$

$$\rho_C^2 = \rho_A^2 + s^2 - 2\rho_A s\cos(\theta_0 - \psi_2) =$$

$$= \rho_A^2 + s^2 - 2\rho_A s\left[\cos\theta_0\cos\left(-\frac{u_0}{2}s + \frac{1}{3}s^2 + O(s^3)\right) + \sin\theta_0\sin\left(-\frac{u_0}{2}s + \frac{1}{3}s^2 + 0(s^3)\right)\right] =$$

$$= \rho_A^2 + s^2 - 2\rho_A s\left[\cos\theta_0\left(1 - \frac{1}{2}\left(-\frac{u_0}{2}s + \frac{1}{3}s^2\right)^2\right) + \sin\theta_0\left(-\frac{u_0}{2}s + \frac{1}{3}s^2\right)\right] =$$

$$= \rho_A^2 - 2\rho_A\cos\theta_0 s + (1 + \rho_A u_0\sin\theta_0)s^2 + \left(\rho_A\frac{u_0^2}{4}\cos\theta_0 - \frac{2}{3}\rho_A\sin\theta_0\right)s^3 + O(s^4);$$

Thus we obtain formula (19):

$$\rho_B^2 - \rho_C^2 = \frac{4}{3}\rho_A\sin\theta_0 s^3 + O(s^4)$$

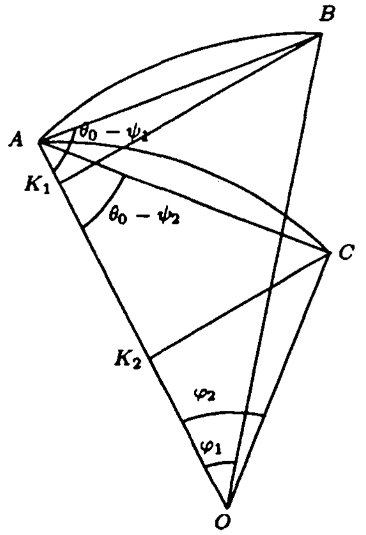



Figure 8

Compute the values of the angles $\delta_1$ and $\delta_2$. From formulas (17) and (18) for the angles $\theta_1$ and $\theta_2$ we obtain:

$$\begin{cases} \theta_1 = \theta_0 + u_0 s + s^2 \\ \theta_2 = \theta_0 + u_0 s - s^2 \end{cases} \tag{22}$$

To compute the angles $\varphi_1$ and $\varphi_2$ make the additional construction (see Figure 8): the segments $BK_1$ and $CK_2$ are perpendicular to the line $OA$. We have

$$|K_1B| = |AB|\sin(\theta_0 - \psi_1) = |OK_1|\tan\varphi_1$$

$$|K_2C| = |AC|\sin(\theta_0 - \psi_2) = |OK_2|\tan\varphi_2$$

Hence

$$\tan\varphi_1 = \frac{|AB|}{|OK_1|}\sin(\theta_0 - \psi_1)$$

$$\tan\varphi_2 = \frac{|AC|}{|OK_2|}\sin(\theta_0 - \psi_2)$$

But

$$|AB| = s + O(s^2), \qquad |AC| = s + O(s^2)$$

$$|OK_1| = |OA| - |AK_1| = \rho_A - |AB|\cos(\theta_0 - \psi_1) = \rho_A - s\cos(\theta_0 - \psi_1)$$

$$|OK_2| = |OA| - |AK_2| = \rho_A - |AC|\cos(\theta_0 - \psi_2) = \rho_A - s\cos(\theta_0 - \psi_2)$$

Thus we have that

$$\tan\varphi_1 = \frac{s\sin(\theta_0-\psi_1)}{\rho_A - s\cos(\theta_0-\psi_1)}; \qquad \tan\varphi_2 = \frac{s\sin(\theta_0-\psi_2)}{\rho_A - s\cos(\theta_0-\psi_2)}$$

Now using formulas (22) and Taylor series for the functions $\cos x$, $\sin x$ and $f(x) = 1/(1+x)$ at 0 we obtain the following expressions:

$$\sin(\theta_0-\psi_1) = \sin\theta_0\cos\psi_1 - \cos\theta_0\sin\psi_1 =$$

$$= \sin\theta_0\left(1-\frac{1}{2}\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right)^2\right)+\cos\theta_0\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right) =$$

$$= \sin\theta_0 + \frac{u_0}{2}\cos\theta_0 s + \left(\frac{\cos\theta_0}{3} - \frac{u_0^2}{8}\sin\theta_0\right)s^2 + O(s^3);$$

$$\cos(\theta_0-\psi_1) = \cos\theta_0\cos\psi_1 + \sin\theta_0\sin\psi_1 =$$

$$= \cos\theta_0\left(1-\frac{1}{2}\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right)^2\right)-\sin\theta_0\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right) =$$

$$= \cos\theta_0 - \frac{u_0}{2}\sin\theta_0 s - \left(\frac{\sin\theta_0}{3} + \frac{u_0^2}{8}\cos\theta_0\right)s^2 + O(s^3);$$

$$\tan\varphi_1 = \frac{s\sin(\theta_0-\psi_1)}{\rho_A}\frac{1}{1-\frac{s}{\rho_A}\cos(\theta_0-\psi_1)} =$$

$$= \frac{s}{\rho_A}\sin(\theta_0-\psi_1)\left(1+\frac{s}{\rho_A}\cos(\theta_0-\psi_1)+\frac{s^2}{{\rho_A}^2}\cos^2(\theta_0-\psi_1)\right)$$

Hence after this series of transformations we obtain the formula for $\tan\varphi_1$:

$$\tan\varphi_1 = \frac{\sin\theta_0}{\rho_A}s + \left(\frac{\sin 2\theta_0}{2\rho_A^2} + \frac{u_0\cos\theta_0}{2\rho_A}\right)s^2 +$$

$$+\left(\frac{\cos\theta_0}{3\rho_A} - \frac{u_0^2\sin\theta_0}{8\rho_A} + \frac{u_0\cos 2\theta_0}{2\rho_A^2} + \frac{\sin 2\theta_0\cos\theta_0}{2\rho_A^3}\right)s^3 + O(s^4) \tag{23}$$

After analogous transformations we obtain the formula for $\tan\varphi_2$:

$$\tan\varphi_2 = \frac{\sin\theta_0}{\rho_A}s + \left(\frac{\sin 2\theta_0}{2\rho_A^2} + \frac{u_0\cos\theta_0}{2\rho_A}\right)s^2 +$$

$$+\left(-\frac{\cos\theta_0}{3\rho_A} - \frac{u_0^2\sin\theta_0}{8\rho_A} + \frac{u_0\cos 2\theta_0}{2\rho_A^2} + \frac{\sin 2\theta_0\cos\theta_0}{2\rho_A^3}\right)s^3 + O(s^4) \tag{24}$$

In a small neighbourhood of the initial point $A$ $\tan\varphi_i = \varphi_i + O(\varphi_i^3)$ $(i = 1, 2)$. Hence, from the definitions of the angles $\delta_1$ and $\delta_2$ and from formulas (23)-(24) we obtain equality (20):

$$\delta_1 - \delta_2 = 2s^2 + \frac{2\cos\theta_0}{3\rho_A}s^3 + O(s^4)$$

The lemma is proved. □

**Proof of Corollary 3.9.**

It follows from (18) that the absolute value of the curvature at the point $C$ is greater than the one at the point $A$. That is why the point $C_c$ is located before the point $A$.

Note that the angles $\gamma_i$ and $\delta_i$ are connected by the following equations: $\gamma_i = \pi - \delta_i$ $(i = 1, 2)$. Hence from formulas (22) and (23) we obtain that

$$\gamma_A - \gamma_B = \delta_1 - \theta_0 = \left(u_0 + \frac{\sin\theta_0}{\rho_A}\right)s + \left(1 + \frac{\sin 2\theta_0}{2\rho_A^2} + \frac{u_0\cos\theta_0}{2\rho_A}\right)s^2 + O(s^3)$$

From Remark 3.5. we obtain that the angle $\gamma$ is a monotonously decreasing function, hence

$$\gamma_B < \gamma_A$$

From (20) we have

$$\gamma_C - \gamma_B = \delta_1 - \delta_2 = 2s^2 + \frac{2\cos\theta_0}{3\rho_A}s^3 + O(s^4)$$

So, $\gamma_B < \gamma_C$ and $\gamma_B < \gamma_A$. But the difference between $\gamma_B$ and $\gamma_A$ is of order $s$, and the difference between $\gamma_B$ and $\gamma_C$ is of order $s^2$. Hence, we obtain the following inequalities

$$\gamma_A > \gamma_C > \gamma_B$$

and the point $C_\gamma$ is located between the points $A$ and $B$.

The difference between $\rho_B^2$ and $\rho_C^2$ is of order $s^3$ (see (19)). The difference between $\gamma_C$ and $\gamma_B$ is of order $s^2$. Hence, the point $C_\rho$ is located between the points $C_\gamma$ and $B$.

The corollary is proved. □

## 3.5 A property of a concatenation of several arcs of clothoids.

Consider two paths with the same initial conditions $(x_0, y_0, \alpha_0, u_0)$ and whose graphs of the curvature as a function of the path length are shown on Figure 9. The path *cl* is a piece of a half-clothoid whose curvature is defined by the equation $u = -2s + u_0$ $(u_0 > 0)$. The path *pcl* consists of several pieces of clothoids whose curvatures are defined by equations of the kind $u = -2s + \tilde{u}^0$ or $u = 2s + \tilde{\tilde{u}}^0$ $(\tilde{u}^0 > 0$ and $\tilde{\tilde{u}}^0 > 0)$, the sum of their lengths is equal to $u_0/2$. Denote by $O_{cl}$ the centre of *cl*, by $\vec{\rho}_{cl}(t)$ the radius-vector of a point of *cl* in the coordinate system with centre at $O_{cl}$. Denote by $\vec{\rho}_{pcl}(t)$ the radius-vector of a point of the path *pcl* in this coordinate system. For $t = 0$ we have $\vec{\rho}_{cl}(0) = \vec{\rho}_{pcl}(0)$.

**Lemma 3.10** *For any path pcl (defined as above) and for the path cl (both paths are defined on the interval $s \in [0, u_0/2]$) we have the following inequality:*

$$\rho_{cl}(s) > \rho_{pcl}(s), \quad \text{for every } s \in (0, u_0/2] \tag{25}$$

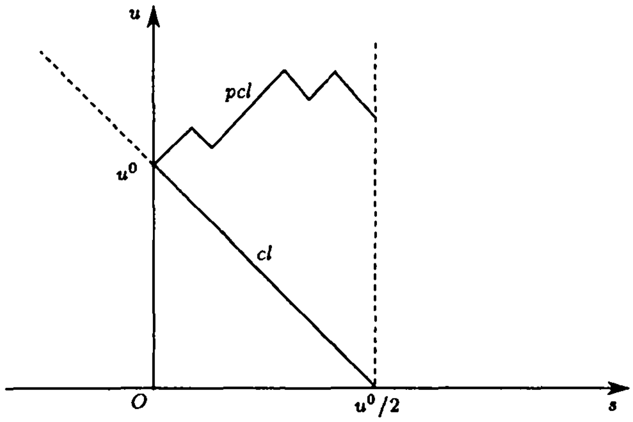



Figure 9

**Proof**

a) Consider the path *cl*. We parametrise it by the natural parameter $s$, setting $s = 0$ for the point $(x_0, y_0, \alpha_0, u_0)$. Hence, the graph of the curvature $u$ as a function of the path length $s$ looks like the one shown on Figure 9 (for $s < 0$ it is given by the dotted line). In the proof we consider the path *cl* only on $[-u_0/2, u_0/2]$.

Denote by $O$ the point of the path *cl* with zero curvature (i.e. $s = u_0/2$), by $A$ - the point with curvature $2u_0$ (i.e. $s = -u_0/2$), by $S$ - the point with curvature $u_0$ (i.e. $s = 0$) and by $P$ - an arbitrary point corresponding to some value of the parameter $s \in (-u_0/2, u_0/2)$ $(u_P(s) \in (0, 2u_0))$, see Figure 10.

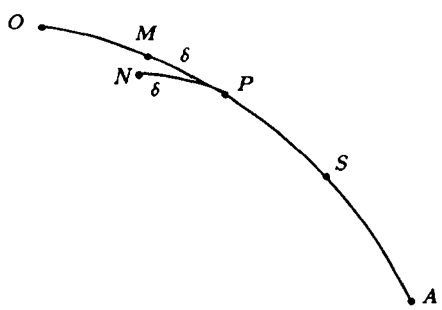



Figure 10

Consider a small $\delta$-half-neighbourhood $(s, s+\delta)$ of the point $P$ and consider a path beginning at the point $P$ which is piecewise clothoid ($u = -2s + \tilde{u}^0$ or $u = 2s + \tilde{\tilde{u}}^0$, $\tilde{u}^0 > 0$, $\tilde{\tilde{u}}^0 > 0$), of length $\delta$ and with the same values of $x, y, \alpha, u$ at the point $P$ as the ones of the point $P$ of *cl*. Denote the final point of this path by $N$, the final point of the corresponding piece of the clothoid *cl* by $M$ (the lengths of the arcs $\widehat{PM}$ and $\widehat{PN}$ are equal to $\delta$, the curvature of *cl* is decreasing from $P$ to $M$). Denote by $N_c$ the point of the clothoid *cl* with the same curvature as the point

$N$, by $N_\rho$ – the point of the clothoid $cl$ with the same length of the radius-vector $\vec{\rho}(t)$ as the point $N$, by $N_\gamma$ – the point of the clothoid $cl$ with the same angle $\gamma(t)$ between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$ as the point $N$. Then for every point $P$ there exists a small $\delta$-half-neighbourhood where the points $N_c, P, N_\gamma, N_\rho, M$ are encountered in this order along $cl$ (see Corollary 3.9). Denote this disposition of the points $N_c, P, N_\gamma, N_\rho, M$ by *disposition**. The number $\delta$ can be chosen the same for all values of $s \in [-u_0/2, u_0/2]$; assume that $\delta$ is fixed.

**b)** Consider some path $\mathcal{P}$ of the class $\mathcal{A}$ of all paths beginning at the point $P$, piecewise clothoid ($u = -2s + \tilde{u}^0$ or $u = 2s + \tilde{\tilde{u}}^0$, $\tilde{u}^0 > 0$, $\tilde{\tilde{u}}^0 > 0$), of length $\leq \nu(s) = u_0/2 - |s|$ and consisting of $n$ pieces ($n > 1/\delta$, each piece being of length $1/n$ except the first one which is of length $\leq 1/n$).

We prove the lemma for paths $\mathcal{P} \in \mathcal{A}$ first, by induction on $n$. For paths $pcl$ defined at the beginning of **3.5** the lemma will be proved in **c)**.

For the first piece of the path $\mathcal{P}$ we have *disposition** (because the length of this piece is $\leq \delta$ and for the $\delta$-half-neighbourhood of the point $P$ we have this disposition). Suppose that *disposition** doesn't hold at some moment $s'$. If $s'$ is the very first moment when it happens, then 3 cases can occur:

1) *If at the moment $s'$ the point $N_\gamma$ coincides with the point $N_\rho$.* Then at the next moment we shall have *disposition**. Really, using the Taylor series, as in Lemma 3.8, we shall obtain the result of Corollary 3.9, because at the moment $s'$ both paths $cl$ and $\mathcal{P}$ have the same value of the radius-vector $\vec{\rho}(t)$ and the same angle $\gamma(t)$ between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$, and the curvature at the point $N_\gamma$ of the path $\mathcal{P}$ is greater than the curvature at the point $N_\gamma$ of the path $cl$.

2) *If at some moment $s'$ the points $N_\gamma$, $N_\rho$ and $N_c$ coincide.* Then this means that we move along a half-clothoid $cl$ but with a delay; hence, we either continue like that and come with a delay, or at some moment we have again *disposition**.

3) *If at some moment $s'$ the point $N_\gamma$ coincides with the point $N_c$ and the point $N_\rho$ is situated after them (see Figure 11).* Then it doesn't happen in the first piece of the path $\mathcal{P}$ (see the definition of $\delta$). Hence, if it happens in the $k$-th piece of the path $\mathcal{P}$ then for the $(k-1)$-st piece of the path $\mathcal{P}$ *disposition** holds. Prove that in this case

$$\rho_{cl}(s) - \rho_{cl}(s') \geq \rho_{\mathcal{P}}(s) - \rho_{\mathcal{P}}(s') \qquad \text{for } s \geq s'$$

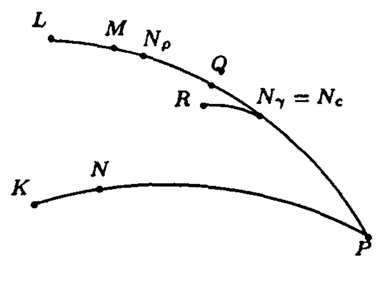



Figure 11

We denote by $M$ the point belonging to the path $cl$ and corresponding to the moment $s'$, by $N$ – the point belonging to the path $\mathcal{P}$ and corresponding to the moment $s'$ (see Figure 11). Note that the notation is the same as the one of Figure 10. Denote by $\widehat{ML}$ an arc of the path $cl$ corresponding to the interval $[s', s' + s^*]$ for some $s^* > 0$ and by $\widehat{NK}$ – an arc of the path $\mathcal{P}$

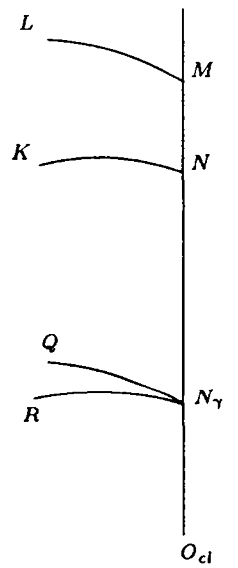
corresponding to the same interval $[s', s' + s^*]$. Denote by $s_{pc}$ ($s_{pc} < s'$) the moment to which the point $N_\gamma = N_c$ corresponds and denote by $\widehat{N_\gamma Q}$ an arc of the path $cl$ corresponding to the interval $[s_{pc}, s_{pc} + s^*]$. Translate the arc $\widehat{NK}$ so that the point $N$ should coincide with the point $N_\gamma$, then rotate the image so that the tangent vector to the image at the point $N_\gamma$ should coincide with the tangent vector to the arc $\widehat{N_\gamma Q}$ at the point $N_\gamma$. Denote the obtained arc by $\widehat{N_\gamma R}$.

For the lengths of the radius-vectors $\vec{\rho}(s)$ at the points $N_\gamma$, $N_p$ and $M$ we have the following inequalities:

$$\rho_{N_\gamma} < \rho_{N_p} < \rho_M$$

This follows from Corollary 3.6 ($\dot{\rho}(s) > 0$).

Rotate the arcs $\widehat{N_\gamma Q}$, $\widehat{N_\gamma R}$ and $\widehat{NK}$ around $O_{cl}$ on different angles so that the points $M$, $N_\gamma$ and $N$ should be on the line $O_{cl}M$, see Figure 12 a).



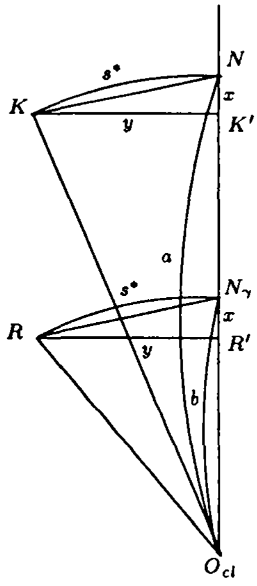




Figure 12

Denote

$$\Delta\rho_{\mathcal{P}} = |\,\overrightarrow{O_{cl}K}\,| - |\,\overrightarrow{O_{cl}N}\,| \qquad \Delta\rho_{\mathcal{P}_{tr}} = |\,\overrightarrow{O_{cl}R}\,| - |\,\overrightarrow{O_{cl}N_\gamma}\,|$$

$$\Delta\rho_{clpr} = |\,\overrightarrow{O_{cl}Q}\,| - |\,\overrightarrow{O_{cl}N_\gamma}\,| \qquad \Delta\rho_{cl} = |\,\overrightarrow{O_{cl}L}\,| - |\,\overrightarrow{O_{cl}M}\,|$$

We know that for the $(k-1)$-st piece of the path $\mathcal{P}$ *disposition** holds. Hence,

$$\Delta\rho_{\mathcal{P}_{tr}} < \Delta\rho_{clpr} \tag{26}$$

(by the inductive assumption, as $k < n$).

Using Corollary 3.6 ($\ddot{\rho} > 0$) we obtain

$$\Delta\rho_{clpr} < \Delta\rho_{cl} \tag{27}$$

Prove that

$$\Delta\rho_{\mathcal{P}} < \Delta\rho_{\mathcal{P}_{tr}} \tag{28}$$

The tangent angles at the points $N$ and $N_\gamma$ are the same, the curvatures – too. Hence (see Figure 12 b)),

$$|KK'| = |RR'| = y; \qquad |NK'| = |N_\gamma R'| = x$$

Denote

$$|O_{cl}N| = a; \qquad |O_{cl}N_\gamma| = b.$$

We have the following equalities:

$$\Delta\rho_{\mathcal{P}} = \sqrt{(a \pm x)^2 + y^2} - a \qquad \Delta\rho_{\mathcal{P}_{tr}} = \sqrt{(b \pm x)^2 + y^2} - b$$

Inequality (28) is equivalent to

$$\sqrt{(a \pm x)^2 + y^2} - a < \sqrt{(b \pm x)^2 + y^2} - b$$

or to

$$\sqrt{(a \pm x)^2 + y^2} - \sqrt{(b \pm x)^2 + y^2} < a - b;$$

$$(a \pm x)^2 - (b \pm x)^2 < (a - b)\left(\sqrt{(a \pm x)^2 + y^2} + \sqrt{(b \pm x)^2 + y^2}\right);$$

$$(a + b \pm 2x) < \left(\sqrt{(a \pm x)^2 + y^2} + \sqrt{(b \pm x)^2 + y^2}\right).$$

Thus we have

$$a + b \pm 2x = (a \pm x) + (b \pm x) \le |a \pm x| + |b \pm x| < \left(\sqrt{(a \pm x)^2 + y^2} + \sqrt{(b \pm x)^2 + y^2}\right)$$

This chain of inequalities is correct, hence, inequality (28) is also correct. Thus, from inequalities (26)–(28) we obtain the desired inequality:

$$\Delta\rho_{\mathcal{P}} < \Delta\rho_{cl}$$

i.e. $\rho_{cl}(s) - \rho_{cl}(s') > \rho_{\mathcal{P}}(s) - \rho_{\mathcal{P}}(s')$ for $s > s'$.

Thus we proved that if at some moment $s'$ *disposition** doesn't hold then for the moments $s > s'$ the length of the radius-vector $\vec{\rho}_{cl}(s)$ for the point belonging to $cl$ is greater than the length of the radius-vector $\vec{\rho}_{\mathcal{P}}(s)$ for the point belonging to $\mathcal{P}$. This holds for any path of the class $\mathcal{A}$ for any point $P$ corresponding to some value of the parameter $s \in (-u_0/2, u_0/2)$.

c) Assume that the point $P$ coincides with the point $S$ (see Figure 10). The curvature of the path $pcl$ and the curvature of any path of the class $\mathcal{A}$ are continuous functions. Hence, if

$n \to \infty$, then we can uniformly approximate the path *pcl* by a sequence of paths of the class $\mathcal{A}$. Hence, for $s \in [0, u_0/2]$ the length of the radius-vector $\vec{\rho}_{cl}(s)$ is greater than the length of the radius-vector $\vec{\rho}_{pcl}(s)$, i.e. inequality (25) is proved.

The lemma is proved. □

Denote by $\mathcal{D}$ the class of the paths with initial conditions $(x_0, y_0, \alpha_0, u_0)$, of length $u_0/2$ and whose graphs of the curvature $u$ as a function of the path length $s$ belong to the class Lip(2). Denote by $\vec{\rho}_L(t)$ the radius-vector of the point of some path $L$ from the class $\mathcal{D}$ in the coordinate system with centre at $O_{cl}$. Then we have

**Corollary 3.11** *For any path $L$ from the class $\mathcal{D}$ and for the path cl from Lemma 3.10 (both paths are defined on the interval $s \in [0, u_0/2]$) we have the following inequality:*

$$\rho_{cl}(s) > \rho_L(s), \quad \text{for every } s \in (0, u_0/2]$$

Really, the class of paths $L$ belongs to the closure of the class of all paths *pcl* defined at the beginning of **3.5**.

# 4 Construction of the suboptimal path.

We construct the suboptimal path in the case when $\mathrm{dist}((x^0, y^0), (x^1, y^1)) \gg 1/\sqrt{B}$ (i.e. there exist constants $a > 1,\ c \geq 0$ such that $\mathrm{dist}((x^0, y^0), (x^1, y^1)) \geq a/\sqrt{B} + c$).

We show that one can construct a path from the initial point $X^0$ with coordinates $(x^0, y^0)$ to the final point $X^1$ with coordinates $(x^1, y^1)$ with four switching points which is a concatenation of four arcs of clothoids and a line segment (along the path the tangent angle and the curvature are continuous, their initial and final values being respectively $\alpha^0, \alpha^1$ and $u^0, u^1$).

Construct the path from $X^0$ to $X^1$ by means of the graph of the curvature $u$ as a function of the path length $s$ (i.e. the natural parameter). Construct at first a part of the path which is a concatenation of two arcs of clothoids only, from the point $X^0$ to some point $X'_D$. For this purpose consider the graph of the curvature $u$ as a function of the path length $s$, which is a piecewise linear and continuous function (the absolute values of the angular coefficients of these pieces are the same, i.e. every piece is of the kind $u = \pm 2s + u_{**}$).

This graph is shown on Figure 13. It is linear on $[0, \xi']$ and on $[\xi', \eta' + 2\xi']$, zero at the point $(\eta' + 2\xi')$. Here $\xi'$ and $\eta'$ are the path lengths, the number $\eta'$ is defined by $u^0$ ($\eta' = 0.5u^0$), $\xi'$ can be considered as a parameter.

Construct the path corresponding to this graph of $u$ from $X^0$ to some point $X'_D$ (the point $X'_D$ of the path corresponds to the point $D$ of the graph $u$).

We can increase the absolute value of the angle $\alpha'$ at the point $X'_D$ by increasing $\xi'$, because the curvature doesn't change sign on $[0, \eta' + 2\xi']$ and the angle $\alpha' - \alpha^0$ is the integral of the curvature on this interval:

$$\alpha' - \alpha^0 = \int_0^{\eta'+2\xi'} u(t)dt$$

Hence, there exist $d'$ and $d'_i$ ($0 \leq d'_i < d'$) such that when $\xi'$ varies in $[d'_i, d']$ the tangent angle $\alpha'$ to the path at the point $X'_D$ assumes continuously all values from $[\pi/2, -\pi/2]\ (\mathrm{mod} 2\pi)$ or $[-\pi/2, \pi/2]\ (\mathrm{mod} 2\pi)$ (we choose the interval depending on the sign of $u^0$).

In conformity with Proposition 3.2 we can take for $d'$ the maximal length of an arc of half-clothoid on which the tangent angle makes a full turn (i.e.$2\pi$).

Estimate the area where the point $X'_D$ can be if $\xi' \in [d'_i, d']$.

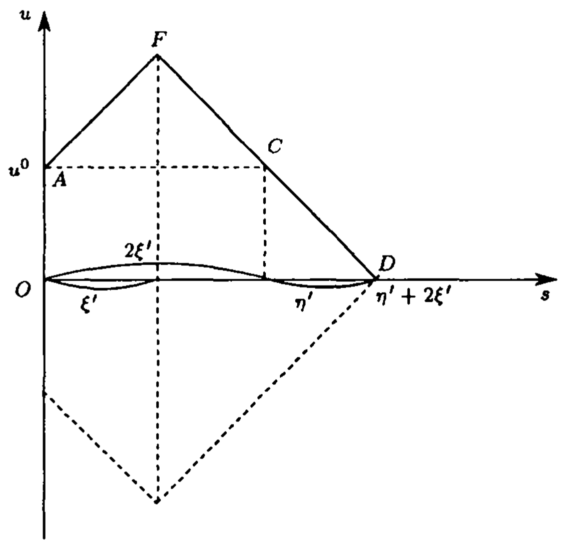



Figure 13

**Proposition 4.1** *If $\xi' \in [d'_i, d']$ the point $X'_D$ will be within some* disc $E'$. *For $x^0$, $\alpha^0$ fixed the coordinates of its centre (which we assume to be the point $X'_D$ for $\xi' = 0$) depend only on $u^0$; its radius doesn't depend on any of the constants $x^0, \alpha^0, u^0$ and when $B$ is not fixed then the radius and the coordinates of the centre depend only on the parameter $B$.*

**Proof**

Consider the circle with centre at the centre of the half-clothoid whose curvature is defined by the part $AF$ of the graph of $u$ as a function of $s$ (see Figure 13). Denote this clothoid by $cl_{in}$. We take the radius of this circle to be equal to the distance between the centre of the half-clothoid and its point with zero curvature (denote it by $r_B$). Denote the point of $cl_{in}$ corresponding to the point $F$ of the graph shown on Figure 13 by $X'_F$. If we change $\xi' \in [d'_i, d']$, then the point $X'_F$ will remain within this circle. The point $X'_D$ will be within the circle with centre at the point $X'_F$ and with radius $r_B$. Thus, the point $X'_D$ will be within the circle $E'$ with centre at the centre of $cl_{in}$ and with radius $2r_B$.

The proposition is proved. □

We can use the same method for constructing the path from $X^1$ to some point $X''_D$ (from the right to the left). For this path we have a parameter $\xi''$, the interval $[d''_i, d'']$ and the disc $E''$ respectively.

Remind that we consider the case when $\text{dist}((x^0, y^0), (x^1, y^1)) \gg 1/\sqrt{B}$. That is why $E' \cap E'' = \emptyset$.

In order to construct the path from $X^0$ to $X^1$ vary $\xi'$ and $\xi''$ so that the tangent lines at the points $X'_D$ and $X''_D$ should be parallel (i.e. $\xi''$ is a function of $\xi'$). For $\alpha' = \pi/2$, $\alpha'' = -\pi/2$ and for $\alpha' = -\pi/2$, $\alpha'' = \pi/2$ the angles between the tangent vector to the path at $X'_D$ and the vector $\overrightarrow{X'_D X''_D}$ have different signs. Hence, if we vary $\xi'$ and $\xi''$ in the intervals $[d'_i, d']$, $[d''_i, d'']$ respectively, for some values $\xi'$, $\xi''$ this angle equal to 0. We obtain the desired path from $X^0$ to $X^1$. The thus constructed path satisfies all the initial requirements.

**Remark 4.2** *We obtain the following inequalities between the radius $r_B$ and the parameters $\xi'$, $\xi''$:*

$$\xi' \leq 2\sqrt{2}r_B$$

$$\xi'' \leq 2\sqrt{2}r_B$$

Really, from formulas (4) we obtain (see Figure 2)

$$r_B = \mid \overrightarrow{OO_c} \mid = \sqrt{x_{O_c}^2 + y_{O_c}^2}$$

where

$$\begin{cases} x_{O_c} = \int_0^\infty \cos(B\tau^2/2)d\tau = \sqrt{2/B}\int_0^\infty \cos\nu^2 d\nu = \sqrt{2/B}\sqrt{2\pi}/4 = \sqrt{\pi}/(2\sqrt{B}) \\ y_{O_c} = \int_0^\infty \sin(B\tau^2/2)d\tau = \sqrt{2/B}\int_0^\infty \sin\nu^2 d\nu = \sqrt{2/B}\sqrt{2\pi}/4 = \sqrt{\pi}/(2\sqrt{B}) \end{cases} \tag{29}$$

Hence

$$r_B = \sqrt{\frac{\pi}{2B}}$$

Remember that $\xi' \in [d_i', d']$ where $d'$ is the maximal length of an arc of a half-clothoid on which the tangent angle to the half-clothoid makes a full turn, $d_i' \geq 0$ (see §4). To compute $d'$ let the point $P_1$ coincide with the point $O$ and let $\alpha_*$ be equal to zero (see Figure 1). Then

$$d' = \widehat{P_1P_3} = \int_0^{\sqrt{4\pi/B}} \sqrt{\cos^2\left(\frac{Bt^2}{2}\right) + \sin^2\left(\frac{Bt^2}{2}\right)}dt = \sqrt{\frac{4\pi}{B}} = 2\sqrt{\frac{\pi}{B}}$$

Hence

$$\xi' \leq 2\sqrt{\frac{\pi}{B}} = 2\sqrt{2}r_B$$

Similarly, $\xi'' \leq 2\sqrt{2}r_B$.

**Remark 4.3** *The initial and final values of the curvature may be positive or negative. That is why the path constructed from $X^0$ to $X^1$ may be of one of the forms shown on Figures 14a)-d). Figure 14a) corresponds to $u^0 > 0$, $u^1 < 0$; Figure 14b) - to $u^0 > 0$, $u^1 > 0$; Figure 14c) - to $u^0 < 0$, $u^1 > 0$ and Figure 14d) - to $u^0 < 0$, $u^1 < 0$. The points $X_D'$, $X_D''$ are the points of zero curvature.*

*It is practically impossible to feel the presence of a switching point between two clothoids on the path (Figures 14a)-d)), because the first and the second derivatives are continuous there. On Figure 15 we show such a switching point - the path $MKL$ contains an arc $(MK)$ of the clothoid $C_1$ and an arc $(KL)$ of the clothoid $C_2$.*

**Remark 4.4** *Consider a path beginning at $X^0$ whose graph of the curvature as a function of the path length has the form shown on Figure 16 ($\zeta > 0$ is a parameter). Such a path will be longer than the path with $\zeta = 0$ (if the tangent angles at $X_D'$ are equal for both paths, the initial angles and curvatures - too).*

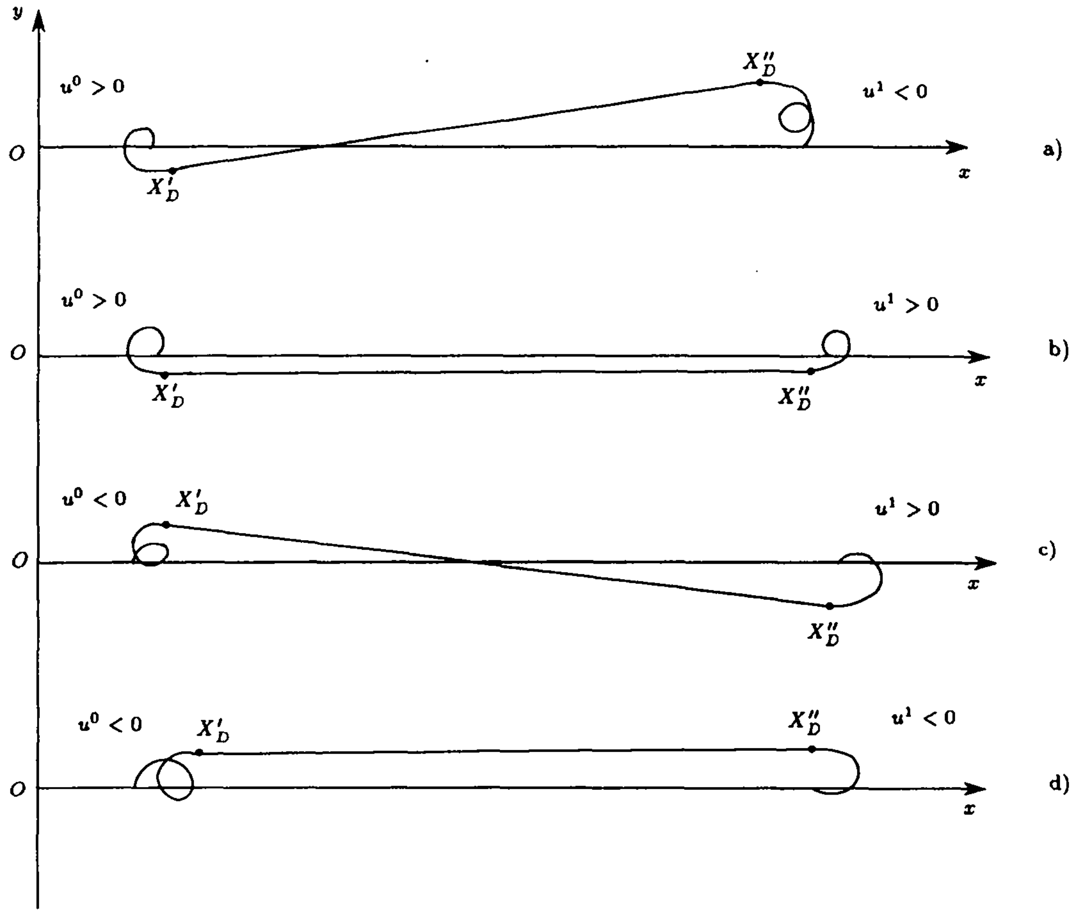



Figure 14

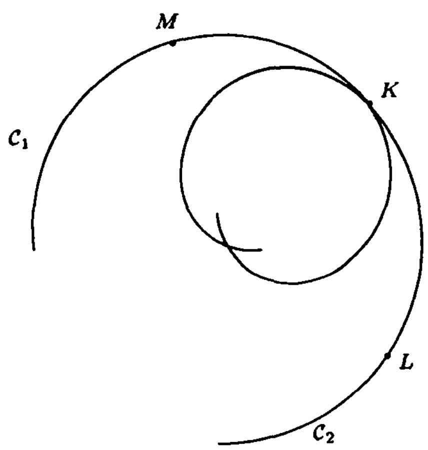



Figure 15

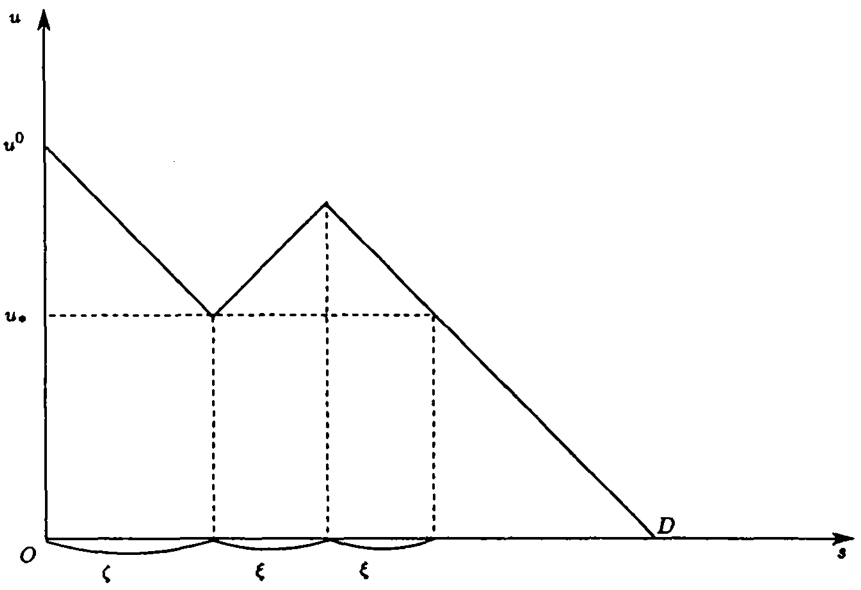



Figure 16

Really, the surfaces under both graphs of the curvature must be equal (because the tangent angle is the integral of the curvature). Hence, $\xi$ is minimal when $u_*$ is maximal, i.e. $\zeta = 0$. This observation makes us choose $\zeta = 0$ for the construction of the suboptimal path.

The condition $\text{dist}((x^0, y^0), (x^1, y^1)) \gg 1/\sqrt{B}$ implies that the line segment between the points $X'_D$ and $X''_D$ is almost horizontal. Hence, if we change $\zeta$ the change of the length $\Delta_l$ of this segment is approximately equal to the change of the length of its projection $\Delta_{l_x}$ on $Ox$. Denote by $\Delta_s$ the change of the total length of the four arcs of clothoid. Denote by $\Delta_{s_x}$ the change of the total length of their projections on $Ox$. Then we have

$$\Delta_s \geq \Delta_{s_x} = -\Delta_{l_x} \approx \Delta_l$$

Therefore one expects to have, in general, shorter paths for smaller values of $\zeta$, because, in general, the left inequality should be strict.

## 5 Proof of the suboptimality of the path constructed in §4.

**Theorem 5.1** *The optimal path for problem (1)–(3) is shorter than the suboptimal path constructed in §4 by no more than* $(10\sqrt{2} + 12)r_B$ *(here* $r_B$ *denotes the distance between the centre of the half-clothoid (4) and its point of zero curvature).*

**Proof.**

$1^0$. Consider the suboptimal path as consisting of five pieces: the first piece is from the initial point $X^0$ to the point $X'_C$ corresponding to the point $C$ on the graph of the curvature $u$ as a function of $s$ (see Figure 13); the second piece is from the point $X'_C$ to the point $X'_D$ (remember that the point $X'_D$ of the path corresponds to the point $D$ on the graph of the curvature $u$); the third piece is a line segment between the points $X'_D$ and $X''_D$; the forth and the fifth pieces are defined in the same way as the second and the first pieces respectively (the point $X''_C$ corresponds to the point $X'_C$).

Consider the initial point $X^0$ with the initial values of the tangent angle and the curvature $\alpha^0$ and $u^0$ as belonging to the unwinding half-clothoid with fixed value of the parameter $B = 2$. Then we can correctly define the centre of this half-clothoid, denoted by $O_{X^0}$. For the final point $X^1$ with $\alpha^1$, $u^1$ we can define the unwinding half-clothoid with centre at the point $O_{X^1}$ respectively.

Then we can consider the optimal path as consisting of three pieces: the first piece is the piece within the circle $D_{X^0}$ with centre at the point $O_{X^0}$ and with radius $r_B$ (more precisely, the piece ends with the first point $P$ which is out of the circle $D_{X^0}$; if the optimal path leaves $D_{X^0}$ and then enters it again, its part after the point $P$ belongs to the second piece). The third piece is the piece within the circle $D_{X^1}$ with centre at the point $O_{X^1}$ and with radius $r_B$ (more precisely, from the last point belonging to $D_{X^1}$ to the point $X^1$). The second piece is what is left between the first and the third one.

**$2^0$**. Remember that we use the folowing notations: we denoted by $X'_F$ the point of the suboptimal path corresponding to the point $F$ on the graph $u$ as a function of $s$ (see Figure 13), by $X'_C$ - the point corresponding to the point $C$, by $X'_D$ - the point corresponding to the point $D$ and by $X''_F$, $X''_C$, $X''_D$ we denoted the points belonging to the corresponding part of the path from the final point.

The point $X'_C$ with $\tilde{\alpha}$, $u_0$ belongs to the unwinding half-clothoid whose centre is correctly defined. Denote it by $O_{X'_C}$. Denote by $D_{X'_C}$ the circle with centre at the point $O_{X'_C}$ and with radius $r_B$.

For the point $X'_C$ we define similarly the point $O_{X''_C}$ and the circle $D_{X''_C}$.

**$3^0$. Plan of the proof of the suboptimality of the path constructed in §4 (the suboptimal path).**

We compare the length of the optimal path and the one constructed in §4. We can estimate the maximal possible difference of their lengths (denote it by $\sigma$). For this purpose we prove that the second (the forth) piece of the suboptimal path is no longer than the first (the third) piece of the optimal one (see **$4^0$**).

Then we estimate the maximal possible length of the pieces $\widehat{X^0X'_C}$ and $\widehat{X^1X''_C}$ of the suboptimal path (see **$5^0$**). Their lengths are, respectively, $2\xi'$ and $2\xi''$.

In **$6^0$** we estimate the maximal possible difference between the distance between the circles defining the second and the forth pieces of the suboptimal one and the distance between the circles defining the first and the third pieces of the optimal one.

And then in **$7^0$** we estimate the difference between the shortest and the longest possible length of the line segment of the suboptimal path.

We summarise these results and obtain $\sigma$ in **$8^0$**.

**$4^0$**. The first and the third pieces of the optimal path belong to the class $\mathcal{D}$ (see the definition in **3.5**). Hence, we obtain from Corollary 3.11 that the second (the forth) piece of the suboptimal path is no longer than the first (the third) piece of the optimal one.

**$5^0$**. We obtain from Remark 4.2 that $\xi' \leq 2\sqrt{2}r_B$ and $\xi'' \leq 2\sqrt{2}r_B$. Hence, adding the pieces $\widehat{X^0X'_C}$ and $\widehat{X^1X''_C}$ we add no more than $2\xi' + 2\xi'' \leq 8\sqrt{2}r_B$ to the length of the suboptimal path.

**$6^0$**. The maximal possible distance between the points $X^0$ and $X'_C$ is equal to $4r_B$, because the arcs $\widehat{X^0X'_F}$ and $\widehat{X'_FX'_C}$ are contained in circles of radius $r_B$. Similarly for the point $X^1$

and $X_C''$. Hence, the maximal possible distance between the points $O_{X^0}$ and $O_{X_C'}$ is equal to $4r_B + r_B + r_B = 6r_B$ (see Figure 17). In the same way the maximal possible distance between the points $O_{X^1}$ and $O_{X_C''}$ is equal to $6r_B$.

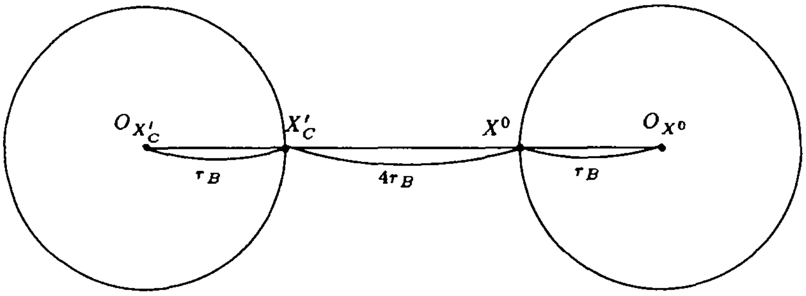



Figure 17

Thus the distance between the circles defining the second and the forth pieces of the suboptimal path is no greater than the distance between the circles defining the first and the third pieces of the optimal one by no more than $6r_B + 6r_B = 12r_B$.

$7^0$. Estimate the difference between the shortest and the longest possible length of the line segment of the suboptimal path. Denote by $RQ$ the line segment of the shortest possible length and by $EW$ the one of the longest possible length (see Figure 18). Denote by $G$ the point belonging to the border of the circle $D_{X_C'}$ and the segment $O_{X_C'}G$ is perpendicular to the line $O_{X_C'}O_{X_C''}$. For the circle $D_{X_C''}$ we have the point $V$ respectively.

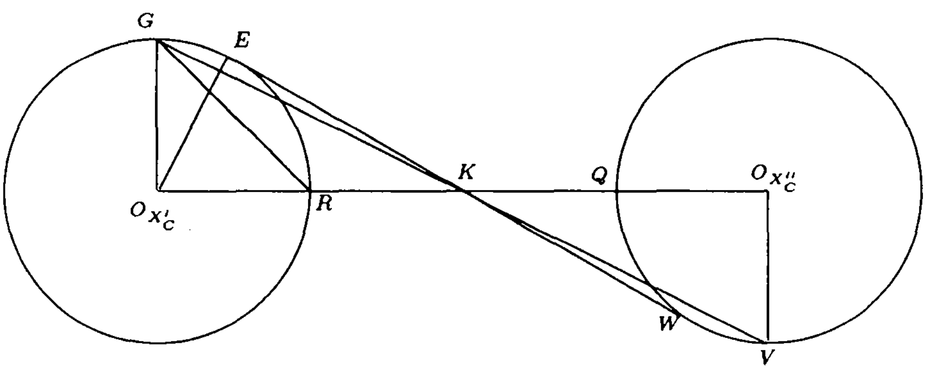



Figure 18

Compute the angle $O_{X_C'}EK$. It is equal to the angle between the vectors $\overrightarrow{OO_c}$ and $\vec{\tau}$ (see Figure 2). The vector $\overrightarrow{OO_c}$ is the radius-vector of the centre $O_c$ of the half-clothoid (5), the vector $\vec{\tau}$ is the tangent vector to this half-clothoid at the point $O$. The line $l$ is perpendicular to the vector $\overrightarrow{OO_c}$ and the angle $\beta$ is the angle between the line $l$ and the tangent vector $\vec{\tau}$. From formulas (29) we obtain that $x_{O_c} = y_{O_c}$, hence, the angle between the axis $Ox$ and the vector $\overrightarrow{OO_c}$ is equal to $\pi/4$ and the angle $\beta$ is equal to $\pi/4$, too. Thus the angle $O_{X_C'}EK$ is equal to $\frac{3}{4}\pi$.

Hence

$$|KE| < |KG|$$

But $|GR| = \sqrt{2}r_B$ (because $|O_{X'_C}G| = |O_{X'_C}R| = r_B$ and $O_{X'_C}G \perp O_{X'_C}R$). Hence,

$$|KE| < |KG| < |GR| + |RK| = \sqrt{2}r_B + |RK|$$

Analogously for the segment $|KW|$ we have the following inequality:

$$|KW| < \sqrt{2}r_B + |KQ|$$

Thus

$$|EW| < |RQ| + 2\sqrt{2}r_B$$

i.e.we obtain that the least possible length of the line segment of the suboptimal path is shorter than the greatest possible length by no more than $2\sqrt{2}r_B$.

$8^0$. Summarising the results obtained in $4^0 - 7^0$, we can estimate the maximal possible difference of the lengths of the suboptimal and the optimal paths:

$$\sigma = 8\sqrt{2}r_B + 12r_B + 2\sqrt{2}r_B = (10\sqrt{2} + 12)r_B$$

The theorem is proved. □

# 6 Acknowledgement

J.D.Boissonnat stated the problem and together with A.Cérézo and J.Leblond participated in numerous and helpful discussions. To all of them we express our most sincere gratitude.

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ISSN 0249 - 6399

*RR_2189*